A NOTE ON INJECTIVITY OF FROBENIUS ON LOCAL COHOMOLOGY OF HYPERSURFACES

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ABSTRACT. Let k be a field of characteristic $p > 0$ such that $[k : k^p] < \infty$ and let $f \in R = k[x_0, \ldots, x_n]$ be homogeneous of degree d. We obtain a sharp bound on the degrees in which the Frobenius action on $H^n_{\mathfrak{m}}(R/fR)$ can be injective when R/fR has an isolated non-F-pure point at $\mathfrak{m}$. As a corollary, we show that if $(R/fR)_{\mathfrak{m}}$ is not F-pure then R/fR has an isolated non-F-pure point at $\mathfrak{m}$ if and only if the Frobenius action is injective in degrees $\leq -n(d-1)$.

1. Introduction

Let k be a field of characteristic p such that $[k : k^p] < \infty$ and let $f \in R = k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d. For simplicity, assume that the test ideal $\tau(f^{1-\frac{1}{p}}) = \mathfrak{m}^j$ for some $j \geq 1$, where $\mathfrak{m} = (x_0, \ldots, x_n)$. Our main theorem obtains the following sharp bound on the degrees in which the Frobenius action on $H^n_{\mathfrak{m}}(R/fR)$ is injective.

Theorem (Theorem 2.14)**.** *If $\tau(f^{1-\frac{1}{p}}) = \mathfrak{m}^j \subseteq \mathfrak{m}$, then the below Frobenius action is injective:*

$$F : H^n_{\mathfrak{m}}(R/fR)_{<-n-j+d} \to H^n_{\mathfrak{m}}(R/fR)_{<p(-n-j+d)}.$$

Our assumption that $\tau(f^{1-\frac{1}{p}}) = \mathfrak{m}^j$ implies that while $(R/fR)_{\mathfrak{m}}$ is not F-pure, $(R/fR)_{\mathfrak{p}}$ is F-pure for every prime $\mathfrak{p} \subsetneq \mathfrak{m}$. We say such rings have an isolated non-F-pure point at $\mathfrak{m}$. The study of F-pure rings has a long history and their theory is rich: Hochster and Roberts first defined F-pure rings and explored the relationship of F-purity to local cohomology (and the Frobenius action thereof) in [HR76]. Fedder continued this program of study, obtaining a criterion for F-purity and showing the equivalence of F-purity and F-injectivity for local Gorenstein rings of characteristic p [Fed83].

A corollary to our main theorem is that when $(R/fR)_{\mathfrak{m}}$ is not F-pure, R/fR has an isolated non-F-pure point at $\mathfrak{m}$ if and only if Frobenius acts injectively in sufficiently negative degrees. Moreover, the degree in which it must be injective depends only on the degree of f.

Theorem (Corollary 2.16)**.** *If $(R/fR)_{\mathfrak{m}}$ is not F-pure then R/fR has an isolated non-F-pure point at $\mathfrak{m}$ if and only if the below Frobenius action is injective:*

$$F : H^n_{\mathfrak{m}}(R/fR)_{\leq -n(d-1)} \to H^n_{\mathfrak{m}}(R/fR)_{\leq -pn(d-1)}.$$

In their study of the F-pure thresholds of Calabi-Yau hypersurfaces, Bhatt and Singh proved a similar result [BS13, Theorem 3.5] under the assumption that R/fR has an isolated singularity at $\mathfrak{m}$. Their methods generalize well to the setting of this paper. The relationship between isolated singularities and isolated non-F-pure points is as follows: regular rings are F-pure, so $\{\text{non-F-pure points of } R/fR\} \subseteq \mathbb{V}(f)_{sing}$. Thus if $(R/fR)_{\mathfrak{m}}$ is not F-pure and has an isolated singularity, it follows that it has an isolated non-F-pure point. Interesting examples of these phenomena often arise as affine cones over smooth projective varieties.

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2. Main result

The *Frobenius map* on a ring A of prime characteristic $p > 0$ is the ring homomorphism $F : A \to A$ given by $F(a) = a^p$. We say that A is F-finite if A is a finitely generated module over $F(A) = A^p$.

We fix notation: throughout, k will denote an F-finite field of characteristic $p > 0$. Let $R = k[x_0, \ldots, x_n]$ be the polynomial ring in $n+1$ variables over k and $f \in R$ be homogeneous of degree d. Note that in this

case R is F-finite. Several of the definitions we provide (2.1, 2.3) rely on F-finiteness for their equivalence to other definitions in the literature. Denote by $\mathfrak{m} = (x_0, \ldots, x_n)$ the homogeneous maximal ideal of R. For an ideal $I \subseteq R$ and a natural number $e \in \mathbb{N}_0$ we denote by $I^{[p^e]} = (u^{p^e} \mid u \in I)$.

We recall a special case of the test ideals introduced by Hara and Yoshida [HY03] and extended to pairs by Takagi [Tak04]. The definition we use is [BMS08, Definition 2.9]; proposition 2.22 from the same paper shows the equivalence of the definitions in the regular F-finite case. The test ideal serves as a positive characteristic analog to the multiplier ideal $\mathscr{J}(X, \mathfrak{a}^t)$ studied in complex algebraic geometry. We refer the reader to [Laz04, Ch. 9 and 10] for an introduction to multiplier ideals.

Definition 2.1 (Test ideal, [BMS08, Definition 2.9])**.** The *test ideal* $\tau(f^{1-\frac{1}{p^e}})$ is the smallest ideal $\mathfrak{a} \subseteq R$ such that

$$f^{p^e-1} \in \mathfrak{a}^{[p^e]}.$$

Remark 2.2. Proposition 2.5 from [BMS08] gives a useful description of $\tau(f^{1-\frac{1}{p^e}})$: let $\{\lambda_b\}_{b\in B}$ be a basis for k over k^{p^e}. The elements $\lambda_b x^{\underline{i}} = \lambda_b x_0^{i_0} \cdots x_n^{i_n}$ with $0 \le i_j \le p^e - 1$ and $b \in B$ form an R^{p^e}-basis for R, so we can express f^{p^e-1} as an R^{p^e}-linear combination

$$f^{p^e-1} = \sum f_{\underline{i},b}^{p^e} \lambda_b x^{\underline{i}}.$$

Then the test ideal $\tau(f^{1-\frac{1}{p^e}})$ is the ideal generated by the $f_{\underline{i},b}$ for all $\underline{i}$ and b appearing above. That is,

$$\tau(f^{1-\frac{1}{p^e}}) = (f_{\underline{i},b} \mid 0 \le i_j \le p^e - 1;\, b \in B).$$

If the Frobenius map $F : A \to A$ is pure, then we say that A is *F-pure.* The corresponding notion in characteristic 0 is that of log canonical (lc) points, and the set of non-lc points is obtained as the vanishing set of the non-lc ideal. Fujino, Schwede, and Takagi initiated development of the theory of *non-F-pure ideals* in [FST11, Section 14]. As one might expect, the vanishing locus of the non-F-pure ideal is precisely the set of primes for which $(R/fR)_\mathfrak{p}$ fails to be F-pure. We caution the reader that the definition we give is specific to the case considered in this note; see [FST11, Definition 14.4] for the general definition.

Definition 2.3 (non-F-pure ideal; [FST11, Remark 16.2])**.** The *non-F-pure ideal* of f, denoted $\sigma(\operatorname{div}(f))$, is defined to be

$$\sigma(\operatorname{div}(f)) = \tau(f^{1-\frac{1}{p^e}}) \text{ for } e \gg 0.$$

Proposition 2.4. $\sqrt{\tau(f^{1-\frac{1}{p}})} = \sqrt{\sigma(\operatorname{div}(f))}$.

Proof. It follows from the definitions that $\sigma(\operatorname{div}(f)) \subseteq \tau(f^{1-\frac{1}{p}})$, so it is enough to show that if $\tau(f^{1-\frac{1}{p}}) \not\subseteq \mathfrak{p}$ for some prime $\mathfrak{p}$ then $\sigma(\operatorname{div}(f)) \not\subseteq \mathfrak{p}$. Since $\sigma(\operatorname{div}(f))$ is the non-F-pure ideal, we check that $(R/fR)_\mathfrak{p}$ is F-pure.

By assumption, $\tau(f^{1-\frac{1}{p}})_\mathfrak{p} = R_\mathfrak{p}$. Since test ideals localize [BMS08, Propostion 2.13(1)] it follows that $f^{p-1} \notin (\mathfrak{p}R_\mathfrak{p})^{[p]}$. Fedder's Criterion [Fed83, Theorem 1.12] now implies that $(R/fR)_\mathfrak{p}$ is F-pure, and so $\sigma(\operatorname{div}(f)) \not\subseteq \mathfrak{p}$. ∎

Definition 2.5 (isolated non-F-pure point)**.** We say that R/fR has an *isolated non-F-pure point* at $\mathfrak{m}$ if $(R/fR)_\mathfrak{m}$ is not F-pure but $(R/fR)_\mathfrak{p}$ is whenever $\mathfrak{p} \subsetneq \mathfrak{m}$.

Remark 2.6. The vanishing set $\mathbb{V}(\sigma(\operatorname{div}(f)))$ is precisely the set of points $\mathfrak{p} \in \mathbb{V}(f)$ such that $(R/fR)_\mathfrak{p}$ is not F-pure. Proposition 2.4 now says that the ideal $\tau(f^{1-\frac{1}{p}})$ also defines this locus. Therefore, R/fR has an isolated non-F-pure point at $\mathfrak{m}$ if and only if $\sqrt{\tau(f^{1-\frac{1}{p}})} = \mathfrak{m}$.

Definition 2.7. Let $e_0 \in \mathbb{N}_0$ be the least integer such that $\tau(f^{1-\frac{1}{p}}) \not\subseteq \mathfrak{m}^{[p^{e_0}]}$. For $e \ge e_0$ define

$$M_e := \min\{\deg(g) \mid g \in (\mathfrak{m}^{[p^e]} : \tau(f^{1-\frac{1}{p}})) \setminus \mathfrak{m}^{[p^e]} \text{ homogeneous}\}.$$

Here we adopt the convention $\min \varnothing = \infty$.

Lemma 2.8. *$M_{e+1} - (n+1)p^{e+1} \leq M_e - (n+1)p^e$ for all $e \geq e_0$.*

Proof. Note that $M_e = \infty$ for $e \geq e_0$ if and only if $\tau(f^{1-\frac{1}{p}}) = R$; in this case there is no content to the lemma. Thus we assume that $M_e < \infty$. For simplicity of notation, write $\tau = \tau(f^{1-\frac{1}{p}})$. Let r be a homogeneous element of $(\mathfrak{m}^{[p^e]} : \tau) \setminus \mathfrak{m}^{[p^e]}$ with minimum degree M_e. Then for each term t of every generator $f_{\underline{i},b}$ for τ (as in Remark 2.2) we have that $\deg_{x_j}(rt) \geq p^e$ for some $0 \leq j \leq n$. Thus,

$$\deg_{x_j}((x_0 \cdots x_n)^{p^{e+1}-p^e} rt) = p^{e+1} - p^e + \deg_{x_j}(rt) \geq p^{e+1}$$

so that $(x_0 \cdots x_n)^{p^{e+1}-p^e} r \in (\mathfrak{m}^{[p^{e+1}]} : \tau)$. Since $(\mathfrak{m}^{[p^{e+1}]} : (x_0 \cdots x_n)^{p^{e+1}-p^e}) = \mathfrak{m}^{[p^e]}$, we know

$$(x_0 \cdots x_n)^{p^{e+1}-p^e} r \notin \mathfrak{m}^{[p^{e+1}]}.$$

It follows that $M_{e+1} \leq M_e + (n+1)(p^{e+1} - p^e)$. ■

Lemma 2.9. *Assume $(R/fR)_{\mathfrak{m}}$ is not F-pure. Then R/fR has an isolated non-F-pure point at $\mathfrak{m}$ if and only if $M_e - (n+1)p^e$ is constant for $e \gg 0$.*

Proof. For simplicity, write $\tau := \tau(f^{1-\frac{1}{p}})$. If $\tau \subseteq \mathfrak{m}$ then $(\mathfrak{m}^{[p^e]} : \tau) \neq \mathfrak{m}^{[p^e]}$ for any e, so $M_e < \infty$ for all e in this case. Since we are assuming $(R/fR)_{\mathfrak{m}}$ is not F-pure, we conclude that $M_e < \infty$ for all e.

R/fR has an isolated non-F-pure point at $\mathfrak{m}$ if and only if $\sqrt{\tau} = \mathfrak{m}$, which is equivalent to $\mathfrak{m}^\ell \subseteq \tau$ for some $\ell \geq 1$.

Claim: $(\mathfrak{m}^{[p^e]} : \tau) \subseteq (\mathfrak{m}^{[p^e]} : \mathfrak{m}^\ell)$ for all $e \gg 0$ if and only if $\mathfrak{m}^\ell \subseteq \tau$.

Proof of claim: Let $(A, \mathfrak{n})$ be a 0-dimensional Gorenstein local ring and let $L \subseteq A$ be an ideal. Write $(-)^\vee$ for the Matlis dual $\operatorname{Hom}_A(-, E_A(A/\mathfrak{n}))$ and note that $A \cong E_A(A/\mathfrak{n})$ since A is 0-dimensional and Gorenstein. Then

$$\begin{aligned}(0 : L) &\cong \operatorname{Hom}_A(A/L, A) \\ &\cong (A/L)^\vee.\end{aligned}$$

Now applying the Matlis dual again, we get $A/L \cong (A/L)^{\vee\vee} \cong (0 : L)^\vee$ where the first isomorphism follows from finite length of A/L. Let $I, J \subseteq A$ be two ideals. If $(0 : J) \subseteq (0 : I)$ then we have an exact sequence

$$0 \to (0 : J) \to (0 : I)$$

which we dualize to get

$$A/I \to A/J \to 0.$$

Thus, if A is a 0-dimensional Gorenstein ring and I, J are two ideals of A then $(0 : J) \subseteq (0 : I)$ if and only if $I \subseteq J$.

Note that $R/\mathfrak{m}^{[p^e]}$ is a 0-dimensional Gorenstein ring for all $e \geq 0$. The above paragraph shows that $(\mathfrak{m}^{[p^e]} : \tau) \subseteq (\mathfrak{m}^{[p^e]} : \mathfrak{m}^\ell)$ if and only if $\mathfrak{m}^\ell + \mathfrak{m}^{[p^e]} \subseteq \tau + \mathfrak{m}^{[p^e]}$. For $e \gg 0$, $\mathfrak{m}^{[p^e]} \subseteq \mathfrak{m}^\ell$ so this last reads $\mathfrak{m}^\ell \subseteq \tau + \mathfrak{m}^{[p^e]}$ for all $e \gg 0$. Therefore

$$\mathfrak{m}^\ell \subseteq \bigcap_{e \gg 0} (\tau + \mathfrak{m}^{[p^e]}).$$

This intersection is τ by Krull's intersection theorem. We conclude that $(\mathfrak{m}^{[p^e]} : \tau) \subseteq (\mathfrak{m}^{[p^e]} : \mathfrak{m}^\ell)$ for $e \gg 0$ if and only if $\mathfrak{m}^\ell \subseteq \tau$. ⊠

The proof of [BS13, Lemma 3.2] shows that

$$(\mathfrak{m}^{[p^e]} : \mathfrak{m}^\ell) = \mathfrak{m}^{[p^e]} + \mathfrak{m}^{(n+1)p^e - n - \ell} \text{ for } e \gg 0.$$

Thus we have that $\sqrt{\tau} = \mathfrak{m}$ if and only if $M_e \geq (n+1)p^e - n - \ell$ for $e \gg 0$ and some $\ell \geq 1$. Lemma 2.8 shows that

$$M_{e+1} - (n+1)p^{e+1} \leq M_e - (n+1)p^e$$

for all $e \geq e_0$, so we conclude that R/fR has an isolated non-F-pure point at $\mathfrak{m}$ if and only if

$$-n - \ell \leq M_e - (n+1)p^e$$

for some $\ell \geq 1$ and all e. Since $\{M_e - (n+1)p^e\}_{e \geq e_0}$ is a nonincreasing sequence of integers, this sequence is bounded below if and only if $M_e - (n+1)p^e$ is constant for $e \gg 0$. ■

Remark 2.10. If $\tau(f^{1-\frac{1}{p}}) = \mathfrak{m}^j$ for some $j \geq 1$ then the proof shows that in fact $M_e - (n+1)p^e = -n - j$ for all $e \geq e_0$.

Remark 2.11. We note that if $M_e < \infty$ then $M_e - (n+1)p^e + d \leq 1 + \frac{d}{p} - \frac{n+1}{p}$. Indeed, if $r \notin \mathfrak{m}^{[p^e]}$ and $\deg(r) = M_e - 1$ then $r \notin (\mathfrak{m}^{[p^e]} : \tau(f^{1-\frac{1}{p}}))$. It follows that $r^p f^{p-1} \notin \mathfrak{m}^{[p^{e+1}]}$. This implies

$$p(M_e - 1) + (p-1)d \leq (n+1)(p^{e+1} - 1).$$

Dividing both sides by p, we have that

$$M_e - (n+1)p^e + d \leq 1 + \frac{d - (n+1)}{p}.$$

In particular, as long as $d \leq n+1$ or $p > d - (n+1)$ we have that $M_e - (n+1)p^e + d \leq 1$.

Definition 2.12. If R/fR has an isolated non-F-pure point at $\mathfrak{m}$, define $\delta(f) = M_e - (n+1)p^e$ for $e \gg 0$.

Of major importance to our proof of the main theorem is analysis of the following diagram of short exact sequences in local cohomology. This appears as [BS13, Remark 2.2].

Remark 2.13. For $f \in R$ as above, the Frobenius map $F : R/fR \to R/fR$ fits into a diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R[-d] & \xrightarrow{f} & R & \longrightarrow & R/fR & \longrightarrow & 0 \\
 & & {\scriptstyle f^{p-1}F}\big\downarrow & & {\scriptstyle F}\big\downarrow & & {\scriptstyle F}\big\downarrow & & \\
0 & \longrightarrow & R[-d] & \xrightarrow{f} & R & \longrightarrow & R/fR & \longrightarrow & 0.
\end{array}$$

The long exact sequence in local cohomology now gives

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^n_{\mathfrak{m}}(R/fR) & \longrightarrow & H^{n+1}_{\mathfrak{m}}(R)[-d] & \xrightarrow{f} & H^{n+1}_{\mathfrak{m}}(R) & \longrightarrow & 0 \\
 & & {\scriptstyle F}\big\downarrow & & {\scriptstyle f^{p-1}F}\big\downarrow & & {\scriptstyle F}\big\downarrow & & \\
0 & \longrightarrow & H^n_{\mathfrak{m}}(R/fR) & \longrightarrow & H^{n+1}_{\mathfrak{m}}(R)[-d] & \xrightarrow{f} & H^{n+1}_{\mathfrak{m}}(R) & \longrightarrow & 0.
\end{array}$$

The rightmost map is injective because R is regular (and so is F-pure), so the snake lemma now implies that injectivity of the map on the left is equivalent to injectivity of the middle map.

Theorem 2.14. *Let $f \in R$ be homogeneous of degree d and assume that R/fR has an isolated non-F-pure point at $\mathfrak{m}$. Then the below Frobenius action is injective:*

$$F : H^n_{\mathfrak{m}}(R/fR)_{<\delta(f)+d} \to H^n_{\mathfrak{m}}(R/fR)_{<p(\delta(f)+d)}.$$

Proof. Writing $N = \delta(f) + d$ we have the diagram in local cohomology

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^n_{\mathfrak{m}}(R/fR)_{<N} & \longrightarrow & H^{n+1}_{\mathfrak{m}}(R)[-d]_{<N} & \xrightarrow{\cdot f} & \cdots \\
 & & {\scriptstyle F}\big\downarrow & & {\scriptstyle f^{p-1}F}\big\downarrow & & \\
0 & \longrightarrow & H^n_{\mathfrak{m}}(R/fR)_{<pN} & \longrightarrow & H^{n+1}_{\mathfrak{m}}(R)[-d]_{<pN+d(p-1)} & \xrightarrow{\cdot f} & \cdots.
\end{array}$$

As remarked above, injectivity of F on the left is equivalent to that of the middle map $f^{p-1}F$. Assume that we have a homogeneous $0 \neq \alpha \in H^{n+1}_{\mathfrak{m}}(R)[-d]_{<N} = H^{n+1}_{\mathfrak{m}}(R)_{<\delta(f)}$ such that $f^{p-1}F(\alpha) = 0$. We have a representation of α of the form

$$\alpha = \left[\frac{g}{(x_0 \cdots x_n)^{p^e}}\right]$$

with $g \notin \mathfrak{m}^{[p^e]}$ and where we may assume that the power in the bottom is p^e for some $e \gg 0$ by multiplying by an appropriate form of 1. Using this representation, we have

$$\begin{aligned} f^{p-1}F(\alpha) = 0 &\iff f^{p-1}g^p \in \mathfrak{m}^{[p^{e+1}]} \\ &\iff f^{p-1} \in \left(\mathfrak{m}^{[p^{e+1}]} : g^p\right) = \left(\mathfrak{m}^{[p^e]} : g\right)^{[p]} \\ &\iff \tau(f^{1-\frac{1}{p}}) \subseteq (\mathfrak{m}^{[p^e]} : g) \\ &\iff g \in \left(\mathfrak{m}^{[p^e]} : \tau(f^{1-\frac{1}{p}})\right). \end{aligned}$$

Here the equality of colon ideals in the second line follows from Kunz's theorem [Kun69, Theorem 2.1] which says Frobenius is flat if and only if R is regular, along with the fact that if $A \to B$ is a flat ring extension then $(I :_A J)B = (IB :_B JB)$ for any ideals $I, J \subseteq A$. Thus, $\deg(g) \geq M_e$ and so

$$\deg(\alpha) = \deg(g) - (n+1)p^e \geq M_e - (n+1)p^e = \delta(f)$$

This contradicts $\deg(\alpha) < \delta(f)$. ■

Remark 2.15. The proof also shows that this bound is optimal: for $e \gg 0$ and an element $r \in (\mathfrak{m}^{[p^e]} : \tau(f^{1-\frac{1}{p}})) \setminus \mathfrak{m}^{[p^e]}$ homogenous of degree M_e, if we take $\alpha = [r/(x_0 \cdots x_n)^{p^e}]$ then $\alpha \neq 0$ but $f^{p-1}F(\alpha) = 0$.

Corollary 2.16. *Let $f \in R$ be homogeneous of degree d and assume that $(R/fR)_{\mathfrak{m}}$ is not F-pure. Then R/fR has an isolated non-F-pure point at $\mathfrak{m}$ if and only if the below Frobenius action is injective:*

$$F : H^n_{\mathfrak{m}}(R/fR)_{\leq -n(d-1)} \to H^n_{\mathfrak{m}}(R/fR)_{\leq -pn(d-1)}.$$

Proof. Assume that R/fR has an isolated non-F-pure point at $\mathfrak{m}$. We show that $-n(d-1) < \delta(f) + d$. As in Remark 2.2, let $\mathscr{F} = \{f_{\underline{i},b} \,|\, 0 \leq i_j \leq p;\, b \in B\}$ be a generating set for $\tau(f^{1-\frac{1}{p}})$. Since R/fR has an isolated non-F-pure point at $\mathfrak{m}$, there exist $n+1$ generators $f_0, \ldots, f_n \in \mathscr{F}$ which form a maximal regular sequence. Write $d_i = \deg(f_i)$. The proof method of [BS13, Lemma 3.1] shows that $\mathfrak{m}^{(\sum d_i)-n} \subseteq (f_0, \ldots, f_n)$. Indeed, let $\mathfrak{b} = (f_0, \ldots, f_n)$. Then the Hilbert series of $R/\mathfrak{b}$ is

$$P(R/\mathfrak{b}, t) = \prod_{i=0}^{n} \frac{1 - t^{d_i}}{1 - t}.$$

This follows from [Eis94, Exercise 21.12(b)] together with the facts that $P(k[x], t) = \frac{1}{1-t}$ and that $P(M \otimes N, t) = P(M, t) \cdot P(N, t)$ whenever all quantities are defined. The degree of this polynomial is $(\sum_{i=0}^n d_i) - (n+1)$. It follows that there can be no monomials of degree greater than $(\sum d_i) - (n+1)$ in $R/\mathfrak{b}$. This is equivalent to $\mathfrak{m}^{(\sum d_i)-(n+1)+1} \subseteq \mathfrak{b}$.

From this we see that $\left(\mathfrak{m}^{[p^e]} : \tau(f^{1-\frac{1}{p}})\right) \subseteq \left(\mathfrak{m}^{[p^e]} : \mathfrak{m}^{(\sum d_i)-n}\right)$ and [BS13, Lemma 3.2] tells us that

$$\left(\mathfrak{m}^{[p^e]} : \mathfrak{m}^{(\sum d_i)-n}\right) = \mathfrak{m}^{[p^e]} + \mathfrak{m}^{(n+1)p^e - (\sum d_i)}.$$

Letting $e \gg 0$ and $r \in \left(\mathfrak{m}^{[p^e]} : \tau(f^{1-\frac{1}{p}})\right) \setminus \mathfrak{m}^{[p^e]}$ be homogeneous of degree M_e, the equality above shows us that

$$\deg(r) = M_e \geq (n+1)p^e - \left(\sum d_i\right).$$

By Lemma 2.9 we now conclude $\delta(f) \geq -(\sum d_i)$. Thus, $\delta(f) + d > -(\sum d_i) + d - 1$. Since $d_i = \deg(f_i)$ we have that $pd_i \leq d(p-1)$ from which it follows that $d_i < d - 1$. Replacing each d_i with $d-1$ we conclude

$$-n(d-1) < -\left(\sum d_i\right) + d - 1 < \delta(f) + d.$$

Using the contrapositive, if R/fR does not have an isolated non-F-pure point at $\mathfrak{m}$, then lemmas 2.8 and 2.9 tell us $\{M_e - (n+1)p^e\}_{e \geq e_0}$ is unbounded below. If $r \in (\mathfrak{m}^{[p^e]} : \tau(f^{1-\frac{1}{p}})) \setminus \mathfrak{m}^{[p^e]}$ has degree M_e then $f^{p-1}F([r/(x_0 \cdots x_n)^{p^e}]) = 0$ but $[r/(x_0 \cdots x_n)^{p^e}] \neq 0$. Letting $e \gg 0$ such that $M_e - (n+1)p^e < -n(d-1)$, we see that the Frobenius action on $H^n_{\mathfrak{m}}(R/fR)_{M_e - (n+1)p^e}$ is not injective. ■

Example 2.17. Let $f = x^2y^2 + y^2z^2 + z^2x^2 \in k[x, y, z]$ with $\mathrm{char}(k) > 2$. Then $\tau(f^{1-\frac{1}{p}}) = \mathfrak{m}$ but f does not have an isolated singularity. In this case, the Bhatt-Singh result [BS13, Theorem 3.5] does not apply. Theorem 2.14 now tells us that the Frobenius action on $H^2_{\mathfrak{m}}(R/fR)$ is injective in degrees ≤ 0. Note that in this case, $H^2_{\mathfrak{m}}(R/fR)_1 \neq 0$ but $H^2_{\mathfrak{m}}(R/fR)_{\geq 2} = 0$ so the Frobenius action on $H^2_{\mathfrak{m}}(R/fR)_1$ is zero.

Example 2.18. We provide an example to show that $M_e - (n+1)p^e$ does not always stabilize at the first step. Let

$$f = x_0^2x_1x_2x_3x_4 + x_0x_1^2x_2x_3x_4 + \cdots + x_0x_1x_2x_3x_4^2 + x_5^6 \in \mathbb{F}_2[x_0, \ldots, x_5].$$

Then $\tau(f^{1/2}) = (x_0, x_1, x_2, x_3, x_4, x_5^3)$. Now $M_1 = 5$, $M_2 = 16$, and we see that $M_1 - 6(2) = -7$ but $M_2 - 6(2^2) = -8$. Since $\mathfrak{m}^3 \subsetneq \tau(f^{1/2})$ we have $-5 - 3 \leq M_e - 6(2^e)$ so we see that $\delta(f) = -8$.

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